MARKOV STATES ON THE CAR ALGEBRA

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ABSTRACT. The program relative to the investigation of quantum Markov states for spin chains based on Canonical Anticommutation Relations algebra is carried on. This analysis provides a further step for a satisfactory theory of quantum Markov processes.

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1. INTRODUCTION

Recently, the investigation of the Markov property in quantum setting had a considerable growth, due to various applications to various fields of Mathematics and quantum Physics. The reader is referred to [4]–[10], [17, 19] and the references cited therein, for recent development of the theory of quantum stochastic processes and their applications.

In some of the above mentioned papers, the Markov property for states of spin algebras on the standard lattice $\mathbb{Z}^d$ is connected with properties of the local Radon–Nikodym density matrices describing the state under consideration and with the Kubo–Martin–Schwinger (KMS for short) boundary condition. Hence, properties of Markov states are related to properties of local Hamiltonians $\{H_\Lambda\}_{\Lambda\subset\mathbb{Z}^d}$ canonically associated to that state.

It should be noted that the usual spin algebra on $\mathbb{Z}^d$ satisfies the commutation rule for observables localized in separated regions.

On the other hand, properties of states on quasi–local algebras satisfying the Canonical Anticommutation Relations (CAR for short) for separated regions are investigated in a sequel of recent papers [11]–[14], that is when Fermion operators are present. The states considered in the last papers are (the generalization of) product states with respect to a fixed partition of the lattice $\mathbb{Z}^d$. Also in this case, connections with local interaction, Hamiltonians and with the KMS property are established.

In the present paper, the program relative to the investigation of quantum Markov states for spin chains based on Canonical Anticommutation Relations algebra is carried on.

As a first step, we restrict the matter to the simplest case when there is only one degree of freedom in each site k of a completely ordered lattice (i.e. when we have a chain which is a subset of $\mathbb{Z}$, and when the CAR algebra in the site k is the full matrix algebra $\mathbb{M}_2(\mathbb{C})$. We then consider non homogeneous states satisfying an appropriate version of the Markov property. In this situation, the structure of such Markov states on CAR algebra is fully understood. Furthermore, the connection of Markov states on on CAR algebra and local Hamiltonians is established.

This analysis provides a first step in order to understand the structure of Markov states on more complicated CAR algebras on the chain, and also on the multidimensional lattice $\mathbb{Z}^d$. It provides also further step towards a satisfactory theory of quantum Markov processes.

Here, for the sake of completeness, we report some preliminary facts which are useful in the sequel.

By a (Umegaki) *conditional expectation* $E : \mathfrak{A} \mapsto \mathfrak{B} \subset \mathfrak{A}$ we mean a norm–one projection of the C^*–algebra $\mathfrak{A}$ onto a C^*–subalgebra (with the same identity I) $\mathfrak{B}$. The map E is automatically a completely positive identity–preserving $\mathfrak{B}$–bimodule map, see [23], Section 9. When $\mathfrak{A}$ is a matrix algebra, the structure of a conditional expectation is well–known, see [20] (see also [18], for more general cases when the centre of the range of E is infinite–dimensional and atomic). Let $\mathfrak{A}$ be a full matrix algebra and consider the (finite) set $\{P_i\}$ of minimal central projections of the range $\mathfrak{B}$ of E, we have

$$E(x) = \sum_i E(P_i x P_i) P_i \,. \tag{1.1}$$

Then E is uniquely determined by its values on the reduced algebras

$$\mathfrak{A}_{P_i} := P_i \mathfrak{A} P_i = N_i \otimes \bar{N}_i$$

where $N_i \sim \mathfrak{B}_{P_i} := \mathfrak{B} P_i$ and $\bar{N}_i \sim \mathfrak{B}'_{P_i} := \mathfrak{B}' P_i$.[1] In fact, there exist states ϕ_i on $\bar{N}_i$ such that

$$E(P_i(a \otimes \bar{a})P_i) = \phi_i(\bar{a}) P_i (a \otimes I) P_i \,. \tag{1.2}$$

Consider a triplet $\mathfrak{C} \subset \mathfrak{B} \subset \mathfrak{A}$ of unital C^*–algebras. A *quasi–conditional expectation* w.r.t. the given triplet, is a completely positive, identity preserving map $E : \mathfrak{A} \mapsto \mathfrak{B}$ such that

$$E(ca) = cE(a) \,, \quad a \in \mathfrak{A} \,, \; c \in \mathfrak{C} \,.\tag{1.3}$$ [2]

[1]The commutant $\mathfrak{B}'$ is considered in the ambient algebra $\mathfrak{A}$.

[2]Notice that, as the quasi–conditional expectation E is a real map, we have

$$E(ac) = E(a)c \,, \quad a \in \mathfrak{A} \,, \; c \in \mathfrak{C}$$

as well.

A pivotal example of quasi–conditional expectation is given by the φ–conditional expectation $E^{\varphi}:\mathfrak{A}\mapsto\mathfrak{B}$ preserving the restriction to the W^*–subalgebra $\mathfrak{B}$ of a normal faithful state φ on the W^*–algebra $\mathfrak{A}$. One can choose for $\mathfrak{C}$ any unital C^*–subalgebra of the W^*–algebra $\mathfrak{B}$ contained in the E^{φ}–fixed point algebra, see [3].

2. Markov States

Let $\mathfrak{A}$ be the *Canonical Anticommutation Relations* (CAR, for short) algebra, with generating element a_i and their adjoints a_i^*, $i\in I$. In our situation, the index set I is a totally ordered countable discrete set I containing, possibly a smallest element j_- and/or a greatest element j_+. Namely, if I contains neither j_-, nor j_+, then $I\sim\mathbb{Z}$. If just $j_+\in I$, then $I\sim\mathbb{Z}_-$, whereas if only $j_-\in I$, then $I\sim\mathbb{Z}_+$. Finally, if both j_- and j_+ belong to I, then I is a finite set and the analysis becomes easier. If I is order–isomorphic to $\mathbb{Z}$, $\mathbb{Z}_-$ or $\mathbb{Z}_+$, we put simbolically j_- and/or j_+ equal to $-\infty$ and/or $+\infty$ respectively. In such a way, the objects with indices j_- and j_+ will be missing in the computations.

The generators $\{a_j,a_j^+\}_{j\in I}$ satisfy the relations

$$\{a_j^+,a_k\}=\delta_{jk}\,,\ \ \{a_j,a_k\}=\{a_j^+,a_k^+\}=0\,,\ \ j,k\in I\,,$$

where $\{\,\cdot\,,\,\cdot\,\}$ stands for the anticommutator. The parity automorphism of $\mathfrak{A}$ is denoted by Θ. For any subset $\Lambda\subset I$, the C^*-subalgebra of $\mathfrak{A}$ generated by a_j,a_j^+ for $j\in\Lambda$ is denoted by $\mathfrak{A}_\Lambda$. It is well–known that $\mathfrak{A}$ is a $\mathbb{Z}_2$–graded algebra, Θ being the grading automorpism. It is well known that the CAR algebra is isomorphic to the C^*–(infinite) tensor product $\overline{\bigotimes_I\mathbb{M}_2(\mathbb{C})}^{C^*}$. For the basic properties of CAR, we refer the reader to [12, 15, 25] and the references cited therein.

In order to avoid technicalities, we deal only with locally faithful states. Furthermore, in order to treat translation invariant or periodic states,[3] we consider only Θ–invariant (quasi–)conditional expectations, if it is not otherwise specified.

Definition 2.1. *A state φ on $\mathfrak{A}$ is called a Markov state if, for each $j_-\leq j<j_+$, there exists a quasi–conditional expectation E_n w.r.t. the triplet $\mathfrak{A}_{n-1]}\subset\mathfrak{A}_{n]}\subset\mathfrak{A}_{n+1]}$ satisfying*

$$\begin{aligned}&\varphi_{n+1]}\circ E_n=\varphi_{n]}\,,\\&E_n(\mathfrak{A}_{[n,n+1]})\subset\mathfrak{A}_{\{n\}}\,.\end{aligned}$$

[3]Relatively to this point, see [12], and [16], Example 5.2.21.

We show that the Markov property defined above can be stated by a sequence of global quasi–conditional expectations, or equally well by sequences ol local or global conditional expectations.

Proposition 2.2. *Let φ be a state on the CAR algebra. The following assertions are equivalent.*

(i) *φ is a Markov state;*
(ii) *the properties listed in Definition 2.1 are satisfied if we replace the quasi–conditional expectations E_n with Umegaki conditional expectations $\mathcal{E}_n$;*
(iii) *for each $j < j_+$, there exists a conditional expectation $\mathcal{E}_{n]} : \mathfrak{A} \mapsto \mathcal{R}(\mathcal{E}_{n]}) \subset \mathfrak{A}_{n]}$ satisfying*

$$\varphi \circ \mathcal{E}_{n]} = \varphi \,,$$

$$\mathcal{E}_{n]}(\mathfrak{A}_{[n}) \subset \mathfrak{A}_{\{n\}} \,; \tag{2.1}$$

(iv) *the properties listed in (iii) are satisfied if we replace the conditional expectations $\mathcal{E}_{n]}$ with quasi–conditional expectations $E_{n]}$*

Proof. It is enough to prove (i)⇒(ii) and (ii)⇒(iii), the remaining implications being trivial.

(i)⇒(ii) Consider the restriction $e_n := E_n\lceil_{\mathfrak{A}_{[n,n+1]}}$ which is a completely positive, identity preserving map leaving invariant a faithful state. Taking the ergodic limit

$$\varepsilon_n := \lim_k \frac{1}{k}\sum_{h=0}^{k-1}(e_n)^h \,,$$

we provide a conditional expectation leaving invariant (the restriction of) the state φ. In order to define a conditional expectation on $\mathfrak{A}_{n+1]}$, we start by noticing that $a \in \mathfrak{A}_{n+1]}$ can be written in a unique way as

$$a = \sum c_{(j_n,j_{n+1})(k_n,k_{n+1})} e(n)_{j_nk_n} e(n+1)_{j_{n+1}k_{n+1}} \,, \tag{2.2}$$

where $c_{(j_n,j_{n+1}),(k_n,k_{n+1})} \in \mathfrak{A}_{n-1]}$, and the $e(n)_{j_nk_n}e(n+1)_{j_{n+1}k_{n+1}}$ given in [25], pag. 92, provide a system of matrix units for $\mathfrak{A}_{[n,n+1]}$. We define for $a \in \mathfrak{A}_{n+1]}$ written as in (2.2),

$$\mathcal{E}_n(a) := \sum c_{(j_n,j_{n+1}),(k_n,k_{n+1})} \varepsilon_n(e(n)_{j_nk_n} e(n)_{j_{n+1}k_{n+1}}) \,.$$

As a generic $a_i \in \mathfrak{A}_{n+1]}$ has the form $a_i = \sum c_{i,\alpha} e_\alpha$, we compute

$$\mathcal{E}_n(a_i^* a_j) = \sum c_{i,\alpha}^* \varepsilon_n(e_\alpha^* e_\beta) c_{j,\alpha} \,,$$

where the equality follows taking into account that ε_n is Θ–invariant. This means that $\mathcal{E}_n$ is completely positive (see [24], Section IV.3). Namely, we

get a norm one projection of $\mathfrak{A}_{n+1]}$ onto a $*$–subalgebra of $\mathfrak{A}_{n]}$ satisfying all property listed in Definition 2.1.
(ii)$\Rightarrow$(iii) Let $m>n$, define

$$\mathcal{E}_{n,m}:=\mathcal{E}_n\circ\cdots\circ\mathcal{E}_{m-1}\,.$$

As $\mathcal{E}_{n,m+k}\lceil_{\mathfrak{A}_{m-1]}}=\mathcal{E}_{n,m}\lceil_{\mathfrak{A}_{m-1]}}$, the direct limit

$$\mathcal{E}^0_n:=\varinjlim_{m\uparrow j_+}\mathcal{E}_{n,m}$$

is a well defined norm one projection of the dense subalgebra $\bigcup\limits_m\mathfrak{A}_{m]}$ onto a subalgebra of $\mathfrak{A}_{n]}$ which, by continuity, uniquely extends to a conditional expectation satisfying the required properties. $\square$

An immediate application of Proposition 2.2 is that the Markov state φ satisfies all the properties listed in Definition 6.1 of [6]. Indeed, it is sufficient to put inside I, $\alpha:=[n+1$, $\bar\alpha=[n$. In such a way, $\alpha'=n]$, $\bar\alpha'=n-1]$ and the projective net of conditional expectations is precisely that formed by the expectations $\mathcal{E}_{n]}$ given in (iii) of Proposition 2.2. However, in order to investigate further structural properties of Markov states when Fermions are present, the Θ–invariance for the conditional expectations and the additional condition (2.1) are needed, see below.

As is stated in Proposition 2.2, the main object is the Θ–invariant two–step conditional expectation ε_n. So, we should describe all Θ–invariant subalgebras of $\mathfrak{A}_{\{n\}}=\mathbb{M}_2(\mathbb{C})$. Of course, $\mathbb{C}I$ and $\mathbb{M}_2(\mathbb{C})$ are trivially Θ–invariant. It remains open the case when the Θ–invariant subalgebra is a maximal abelian subalgebra of $\mathbb{M}_2(\mathbb{C})$.

Lemma 2.3. *The unique Θ–invariant maximal abelian subalgebra of the CAR algebra generated by a,a^+ is generated by the projection aa^+ and a^+a.*

Proof. Let P one of the minimal projection generating the algebra under consideration. Then $P=P_++P_-$ is its splitting in the even and odd part. Then, $\Theta(P)$ is another minimal projection in the same maximal abelian subalgebra. This means that $\Theta(P)=I-P$, which is excluded as this implies $I=2P_+$. The remaining possibility is $\Theta(P)=P$, which is equivalent to $P=P_+$. The last assertion turns out to be equivalent to $P=aa^+$, or $P=a^+a$ which is the assertion. $\square$

Lemma 2.4. *If $\mathcal{R}(\varepsilon_n)=\mathfrak{A}_{\{n\}}$ then*

$$\varepsilon_n(\mathfrak{A}_{\{n+1\},-})=0\,.$$

Proof. If $x_{n+1} \in \mathfrak{A}_{\{n+1\}}$ is odd, then x_{n+1} anticommutes with a_n, a_n^+. Hence, $\varepsilon_n(x_{n+1})$ anticommutes with a_n, a_n^+ as well. As

$$\varepsilon_n(x_{n+1}) = \alpha a_n^+ + \beta a_n + \gamma a_n a_n^+ + \delta a_n^+ a_n \,,$$

we have

$$a_n \varepsilon_n(x_{n+1}) = \alpha a_n a_n^+ + \gamma a_n \,,$$
$$\varepsilon_n(x_{n+1}) a_n = \alpha a_n^+ a_n + \delta a_n \,.$$

Using the above anticomutation properties, we infer that

$$\alpha(a_n a_n^+ + a_n^+ a_n) + (\gamma + \delta) a_n = 0,$$

which implies $\alpha = 0$ and $\delta = -\gamma$.

By the similar argument applied to a_n^+, we get $\beta = 0$. Thus,

$$\varepsilon_n(x_{n+1}) = \gamma(a_n^+ a_n - a_n a_n^+)$$

which means $\gamma = 0$ as ε is supposed to be Θ–invariant. □

We pass to the study of the structure of the ε_n for the three possibilities listed below (see Lemma 2.3). This should be done with some caution, as (1.1), (1.2) do not directly apply to our situation. According to the standard terminology reported in pag. 92 of [25], we put

$$P_1^n := a_n a_n^+ \equiv e_{11}(n) \,,$$
$$P_2^n := a_n^+ a_n \equiv e_{22}(n) \,.$$

Proposition 2.5. *Under the above assumptions, the following assertions hold true.*

(i) *If $\mathcal{R}(\varepsilon_n) = \mathbb{C}I$, then there exists a even state Φ_n on $\mathfrak{A}_{[n,n+1]}$ such that $\varepsilon_n(x) = \Phi_n(x)I$;*

(ii) *If $\mathcal{R}(\varepsilon_n) = \mathfrak{A}_{\{n\},+}$ then there exist even states Φ_1^n, Φ_2^n on $\mathfrak{A}_{\{n+1\}}$ such that, for $x \in \mathfrak{A}_{\{n\}}$, $y \in \mathfrak{A}_{\{n+1\}}$,*

$$\varepsilon_n(xy) = \mathrm{Tr}(xP_1^n)\Phi_1^n(y)P_1^n + \mathrm{Tr}(xP_2^n)\Phi_2^n(y)P_1^n \,;$$

(iii) *$\mathcal{R}(\varepsilon_n) = \mathfrak{A}_{\{n\}}$ then there exists a even state Ψ_n on $\mathfrak{A}_{\{n+1\}}$ such that, for $x \in \mathfrak{A}_{\{n\}}$, $y \in \mathfrak{A}_{\{n+1\}}$, $\varepsilon_n(xy) = x\Psi_n(y)$.*

Proof. (i) and (ii) easily follow by (1.1), (1.2), taking into account that $\mathfrak{A}_{\{n\},+} \bigvee \mathfrak{A}_{\{n+1\}} \sim \mathfrak{A}_{\{n\},+} \otimes \mathfrak{A}_{\{n+1\}}$ ([25], pag. 94), and Θ–invariance of ε_n.

(iii) By a repeated application of Lemma 2.4, if $x \in \mathfrak{A}_{\{n\}}$, $y \in \mathfrak{A}_{\{n+1\}}$, we have

$$x\varepsilon_n(y) = x\varepsilon_n(y_+) = \varepsilon_n(xy_+) = \varepsilon_n(y_+x) = \varepsilon_n(y_+)x = \varepsilon_n(y)x \,.$$

This means that $\varepsilon_n(y) \in \mathcal{Z}(\mathfrak{A}_{\{n\}}) \equiv \mathbb{C}I$. The assertion follows again by Θ–invariance of ε_n. □

It is immediate to verify that Φ_n, Ψ_n are the restrictions of φ to $\mathfrak{A}_{[n,n+1]}$, $\mathfrak{A}_{\{n+1\}}$ respectively, and

$$\Phi_i^n = \frac{\varphi(P_i^n \cdot)}{\varphi(P_i^n)}\,, \quad i = 1, 2\,.$$

We leave to the reader the proof of the following

Lemma 2.6. *Let φ be a Markov state on the CAR algebra, and $\{\varepsilon_j\}_{j_-\leq j<j_+}$ the associated sequence of two–point conditional expectations. Then*

$$\varphi(x_k \cdots x_l) = \varphi((\varepsilon_k(x_k\varepsilon_{k+1}(x_{k+1}\cdots\varepsilon_{l-1}(x_{l-1}x_l)\cdots))) \tag{2.3}$$

for every $k, l \in I$ with $k < l$, and $x_k x_{k+1}\cdots x_{l-1}x_l$ any linear generator of $\mathfrak{A}_{[k,l]}$.

Now we show that a Markov state can be obtained by lifting, via a suitable conditional expectation, its restriction to a subalgebra. This property is analogous to the corresponding one for Markov states on spin chains (see [9]), and plays a crucial rôle in the sequel (see also [4], Section 3).

We start by defining a conditional expectation onto a subalgebra of the CAR algebra $\mathfrak{A}$. Let $\Gamma \subset I\backslash\{j_+\}$ be defined as the set of sites n such that $\mathcal{R}(\varepsilon_n) = \mathfrak{A}_{\{n\},+}$. Define $\mathcal{E} : \mathfrak{A} \mapsto \overline{\mathfrak{A}_{I\backslash\Gamma}\bigvee\big(\bigvee_{n\in\Gamma}\mathfrak{A}_{\{n\},+}\big)}^{C^*}$ as follows. Put

$$\mathcal{E} := \prod_{j\in I} F_j\,, \tag{2.4}$$

where F_j is the identity if $j \notin \Gamma$, and

$$F_j(x) = P_1^j x P_1^j + P_2^j x P_2^j$$

otherwise. The map $\mathcal{E}$ is well defined on localized elements and extends by continuity to a conditional expectation on $\mathfrak{A}$ onto $\overline{\mathfrak{A}_{I\backslash\Gamma}\bigvee\big(\bigvee_{n\in\Gamma}\mathfrak{A}_{\{n\},+}\big)}^{C^*}$.

Proposition 2.7. *Let φ be a Markov state, and $\mathcal{E}$ the conditional expectation defined in* (2.4)*. Then $\varphi = \varphi\circ\mathcal{E}$.*

Proof. Taking into account (1.1), we get if $n \in \Gamma$,

$$\varepsilon_n(xy) = \sum_{k=1}^{2}\varepsilon_n(P_k^n xyP_k^n)P_k^n) = \varepsilon_n\bigg(\sum_{k=1}^{2}P_k^n xP_k^n y\bigg) = \varepsilon_n(\mathcal{E}(x)y)\,.$$

Hence, by Lemma 2.6 we obtain for $k < l < j_+$, and x_j linear generators of $\mathfrak{A}$,

$$\begin{aligned}&\varphi(x_k\cdots x_l) = \varphi((\varepsilon_k(x_k\varepsilon_{k+1}(x_{k+1}\cdots\varepsilon_l(x_l)\cdots)))\\ &= \varphi((\varepsilon_k(\mathcal{E}(x_k)\varepsilon_{k+1}(\mathcal{E}(x_{k+1})\cdots\varepsilon_l(\mathcal{E}(x_l)\cdots))) = \varphi(\mathcal{E}(x_k)\cdots\mathcal{E}(x_l))\end{aligned}$$

which leads to the assertion. $\square$

3. THE STRUCTURE OF MARKOV STATES

In this section we provide a disintegration of a Markov state into elementary Markov states. This allows us to give a reconstruction theorem. These results parallels the analogous one described in [4].

We start by partitioning $I\backslash\{j_+\}$ into disjoint intervals each of which consisting of points n such that $\mathcal{R}(\varepsilon_n)$ is trivial (i.e. $\mathbb{C}I$ or $\mathfrak{A}_{\{n\}}$), or $\mathcal{R}(\varepsilon_n) = \mathfrak{A}_{\{n\},+}$. In this way, $\Gamma = \mathring{\bigcup}_k \Gamma_k$ (where $\mathring{\cup}$ stands for disjoint union), and $\Gamma_k = (l_k - 1, r_k + 1)$.

Define

$$\Omega := \prod_k \Omega_k\,, \quad \Omega_k := \prod_{l_k-1<j<r_k+1} \{1,2\}\,, \quad \mu := \prod_k \mu_k\,, \tag{3.1}$$

where μ_k is the Markov measure on Ω_k determined by the distributions $\pi^j_{\omega_j}$ at place j and the transition coefficients $\pi^j_{\omega_j\omega_{j+1}}$ given by

$$\begin{aligned} \pi^j_{\omega_j} &= \varphi(P^j_{\omega_j})\,, \quad l_k - 1 < j < r_k + 1\,,\ \omega_j = 1,2\,, \\ \pi^j_{\omega_j\omega_{j+1}} &= \frac{\varphi(P^j_{\omega_j}P^{j+1}_{\omega_{j+1}})}{\varphi(P^j_{\omega_j})}\,, \quad l_k - 1 < j < r_k\,,\ \omega_j, \omega_{j+1} = 1,2\,. \end{aligned} \tag{3.2}$$

Notice that the range of $\mathcal{E}$ given in (2.4) can be described by the C^*–algebra consisting of all continuous functions $\omega \in \Omega \mapsto x(\omega) \in \mathfrak{A}_{I\backslash\Gamma}$. Furthermore, the measure μ is precisely given, under a standard isomorphism, by the restriction of the Markov state φ to the Abelian C^*–subalgebra generated by the projections $\{P^j_{\omega_j} \,\big|\, j \in \Gamma\,, \omega_j = 1,2\}$.

Starting from the Markov state φ, consider, for $\omega \in \Omega$ the product state extension (product state for short, see [13])

$$\psi_\omega = \prod_k \psi_{k,\omega}\,, \tag{3.3}$$

on $\mathfrak{A}_{I\backslash\Gamma}$. Here, $\psi_{k,\omega}$ is the one–step or two–step product state on $\mathfrak{A}_{(r_k,l_{k+1})}$ depending only on $\omega_{r_k}, \omega_{l_{k+1}}$, constructed as follows.[4]

(i) If $k+1$ is the first element of Γ not equal to j_-, or $\mathcal{R}(\varepsilon_{r_k+1}) = \mathbb{C}I$, then

$$\psi_{k,\omega}(x) := \frac{\varphi(xP^{l_{k+1}}_{\omega_{l_{k+1}}})}{\varphi(P^{l_{k+1}}_{\omega_{l_{k+1}}})}\,,$$

[4]If $k+1$ is the first element of Γ not equal to j_-, then $r_k = j_-$. If k is the last element of Γ, then $l_{k+1} = j_+$. We are using also intervals without the boundary elements in order to take into account the possibility of $j_- = -\infty$ and/or $j_+ = +\infty$.

(ii) if k is the last element of Γ, or $\mathcal{R}(\varepsilon_{l_{k+1}-1})=\mathfrak{A}_{\{l_{k+1}-1\}}$, then

$$\psi_{k,\omega}(x):=\frac{\varphi(P^{r_k}_{\omega_{r_k}}x)}{\varphi(P^{r_k}_{\omega_{r_k}})}\,,$$

(iii) if the interval under consideration has on the left and on the right, elements of Γ, that is it has the form $[r_k+1,l_{k+1}-1]$, then

$$\psi_{k,\omega}(x):=\frac{\varphi(P^{r_k}_{\omega_{r_k}}xP^{l_{k+1}}_{\omega_{l_{k+1}}})}{\varphi(P^{r_k}_{\omega_{r_k}})\varphi(P^{l_{k+1}}_{\omega_{l_{k+1}}})}\,,$$

(iv) for $r_k<j<l_{k+1}-1$, the two–step factor $\psi_{k,\omega}(x)$, $x\in\mathfrak{A}_{[j,j+1]}$ appears iff $\mathcal{R}(\varepsilon_j)=\mathbb{C}I$ and $\mathcal{R}(\varepsilon_{j+1})=\mathfrak{A}_{\{j+1\}}$.

Notice that, by Proposition 2.5, the states ψ_ω are even. Finally, it is easy to show that the map

$$\omega\in\Omega\mapsto\psi_\omega\in\mathcal{S}(\mathcal{R}(\mathcal{E})) \tag{3.4}$$

is measurable in the weak–$*$ topology.

We are ready to prove a result concerning the decomposition of a Markov state on the CAR algebra into elementary Markov states following the strategy developed in Section 3 of [4].

Theorem 3.1. *Let φ be a Markov state on the CAR algebra $\mathfrak{A}$.*

Then φ admits a direct–integral decomposition

$$\varphi=\int^{\oplus}_{\Omega}\psi_\omega(\mathcal{E}(\,\cdot\,)(\omega))\mu(\mathrm{d}\omega)\,, \tag{3.5}$$

where the measure space (Ω,μ) is defined in (3.1), (3.2), *the conditional expectation $\mathcal{E}$ is given in* (2.4), *the state ψ_ω is given in* (3.3) *through (i)–(iv) above, and finally the integral* (3.5) *is understood as a L^1–direct integral.*[5]

Proof. We sketch the proof which is quite similar to that of Theorem 3.2 of [4].

Put $\mathfrak{B}:=\mathcal{R}(\mathcal{E})$. Consider the abelian C^*–subalgebra $\mathfrak{Z}$ of $\mathfrak{B}$ generated by P^j_i, $j_-\leq j<j_+$, $i=1,2$, together with the GNS representation π of $\mathfrak{B}$ relative to $\varphi\lceil_{\mathfrak{B}}$. Then $\pi(\mathfrak{Z})''\subset\pi(\mathfrak{B})'\cap\pi(\mathfrak{B})''$. As $\pi(\mathfrak{Z})''\sim L^\infty(\Omega,\mu)$, we have for π the direct–integral disintegration

$$\pi=\int^{\oplus}_{\Omega}\pi_\omega\mu(\mathrm{d}\omega)$$

where $\omega\mapsto\pi_\omega$ is a weakly measurable field of representations of $\mathfrak{B}$, see [24], Theorem IV.8.25.

[5] See [24], Section IV.8.

Further, by mimicking the proof of Proposition IV.8.34 of [24], we find a measurable field $\omega\mapsto\xi_\omega$ of vectors such that, for each $x\in\mathfrak{B}$, we get

$$\varphi(x)=\int_\Omega^\oplus\langle\pi_\omega(x)\xi_\omega,\xi_\omega\rangle\mu(\mathrm{d}\omega)\,.$$

As φ is a Markov state, it is invariant w.r.t. $\mathcal{E}$. Then

$$\varphi=\int_\Omega^\oplus\varphi_\omega\mu(\mathrm{d}\omega)$$

for the measurable field φ_ω defined as

$$\varphi_\omega:=\langle\pi_\omega(\mathcal{E}(\,\cdot\,))\xi_\omega,\xi_\omega\rangle\,.$$

Fix localized elements $x\in\mathfrak{A}$, $z\in\mathfrak{Z}\sim C(\Omega)$. It is easy to show by applying the Markov property, that

$$\int_\Omega z(\omega)\varphi_\omega(x)\mu(\mathrm{d}\omega)=\int_\Omega z(\omega)\psi_\omega(E_\omega(x))\mu(\mathrm{d}\omega)$$

for each fixed localized operator $x\in\mathfrak{A}$ and each $z\in\mathfrak{Z}$ represented by the function $z(\omega)$ in $C(\Omega)$ depending only on finitely many variables. As such functions are dense in $C(\Omega)$, we conclude by the uniqueness of the Radon–Nikodym derivative, that for each localized element $x\in\mathfrak{A}$, there exists a measurable set $\Omega_x\subset\Omega_0$ of full μ–measure such that, when $\omega\in\Omega_x$, we have,

$$\varphi_\omega(x)=\psi_\omega(E_\omega(x))\,. \tag{3.6}$$

By considering linear combinations with rational coefficients, we can select a measurable set $F\subset\Omega_0$ of full μ–measure and a dense subset $\mathfrak{A}_0\subset\mathfrak{A}$ of localized operators such that (3.6) continues to be true on F, for each element of $\mathfrak{A}_0$.

Consider now a sequence $x_n\in\mathfrak{A}_0$ converging to $x\in\mathfrak{A}$. If $\omega\in F$ we obtain

$$\varphi_\omega(x)=\lim_n\varphi_\omega(x_n)=\lim_n\psi_\omega(E_\omega(x_n))=\psi_\omega(E_\omega(x))\,,$$

that is (3.6) holds on $F\subset\Omega_0$, simultaneously for each $a\in\mathfrak{A}$. □

An immediate consequence of Theorem 3.1 and Proposition 2.5, is that, according to our assumptions, a Markov state is automatically even.

We pass to a reconstruction theorem which parallels the analogous one in [4].

We start by choosing a subset $\Gamma\subset I\backslash\{j_+\}$ together with a classical Markov process on Ω given in (3.1) with the Markov measure μ_k on Ω_k determined by the distributions $\pi^j_{\omega_j}$ at place j and the transition matrices $\pi^j_{\omega_j\omega_{j+1}}$. For each ω, form, according to the prescription (iv) above, an even one–step or two–step product state ψ_ω on $\mathfrak{A}_{I\backslash\Gamma}$ depending only on the

boundaries ω_{r_k}, $\omega_{l_{k+1}}$, of the decomposition of Γ into connected intervals (as before, the subscript k describes such a decomposition). Such states are well defined, taking into account Theorem 1 of [13]. Moreover, the map given as in (3.4) is measurable in the weak–$*$ topology.

Define $\psi\in\mathcal{S}(\mathfrak{A})$ as

$$\psi:=\int_\Omega^\oplus\psi_\omega(\mathcal{E}(\,\cdot\,)(\omega))\mu(\mathrm{d}\omega)\,. \tag{3.7}$$

Consider, for each $n\in I\backslash\{j_+\}$ the Θ–invariant conditional expectation

$$\mathcal{E}_n:\mathfrak{A}_{n+1]}\mapsto\mathcal{R}(\mathcal{E}_n)\subset\mathfrak{A}_{n]}$$

uniquely determined by setting for $x\in\mathfrak{A}_{n-1]}$, $x_n\in\mathfrak{A}_{\{n\}}$, $x_{n+1}\in\mathfrak{A}_{\{n+1\}}$,

$$\mathcal{E}_n(xx_nx_{n+1}):=x\psi(x_nx_{n+1})$$

if the two–step factor $\psi(x_nx_{n+1})$ appears in the decomposition of ψ, or $n=l_k-1$ (l_k being the left boundary of some interval of Γ) and ψ_ω depends on ω_{l_k};

$$\mathcal{E}_n(xx_nx_{n+1}):=xx_n\psi(x_{n+1})$$

if the one–step factor $\psi(x_{n+1})$ appears in the decomposition of ψ, or $n=r_k+1$ (r_k being the right boundary of some interval of Γ) and ψ_ω depends on ω_{r_k};

$$\mathcal{E}_n(xx_nx_{n+1}):=x\sum_{\omega_n=1}^{2}\mathrm{Tr}_{\,\mathfrak{A}_{\{n\}}}(x_nP^n_{\omega_n})\frac{\psi(P^n_{\omega_n}x_{n+1})}{\psi(P^n_{\omega_n})}P^n_{\omega_n}$$

if $n\in\Gamma$.

Theorem 3.2. *Let $\psi\in\mathcal{S}(\mathfrak{A})$ in* (3.7) *be constructed by the prescriptions listed above. Then it is a Markov state w.r.t. the sequences $\{\mathcal{E}_n\}_{j_-\leq n<j_+}$ of the above mentioned conditional expectations.*

Proof. A straighforward computation, taking into account the various possibilities, see the analogous proof of Theorem 4.1 of [4]. $\square$

4. CONNECTION WITH STATISTICAL MECHANICS

In this section we investigate natural connections between the Markov property and the KMS conditions for states on CAR algebra. This provides natural applications to quantum statistical mechanics, see [4]–[12], for other analogous connections.

Suppose we have a locally faithful state on the CAR algebra $\mathfrak{A}$, then a potential h_Λ is canonically defined for each finite subset Λ of the index set I by

$$\varphi_{\lceil\mathfrak{A}_\Lambda}=\mathrm{Tr}_{\,\mathfrak{A}_\Lambda}(e^{-h_\Lambda}\,\cdot\,)\,. \tag{4.1}$$

Such a set of potentials $\{h_\Lambda\}_{\Lambda\subset I}$ satisfies normalization conditions

$$\mathrm{Tr}\,_{\mathfrak{A}_\Lambda}(e^{-h_\Lambda})=1\,,$$

together with compatibility conditions

$$(\mathrm{Tr}\,_{\mathfrak{B}_\Lambda}\otimes\mathrm{id}\,_{\mathfrak{A}_\Lambda})(e^{-h_{\widehat\Lambda}})=e^{-h_\Lambda}$$

for finite subsets $\Lambda\subset\widehat\Lambda$, whenever $\mathfrak{A}_{\widehat\Lambda}\cong\mathfrak{B}_\Lambda\otimes\mathfrak{A}_\Lambda$.

As the structure of Markov states is fully understood, the set of potentials related to φ by (4.1) can be esplicitely written and satisfies some nice properties.

We start by defining sequences of selfadjoint matrices $\{H_j\}_{j_-\leq j\leq j_+}$, $\{\widehat H_j\}_{j_-\leq j\leq j_+}$ localized in $\mathfrak{A}_{\{j\}}$, and $\{H_{j,j+1}\}_{j_-\leq j<j_+}$ localized in $\mathfrak{A}_{[j,j+1]}$ respectively. Let the distribution $\pi^j_{\omega_j}$ at place j, and the transition coefficients $\pi^j_{\omega_j\omega_{j+1}}$ be defined in (3.2). Denote by ρ_ψ the density–matrix associated to a strictly positive functional ψ on a full matrix algebra. If $\varepsilon_j=\mathfrak{A}_{\{j\},+}$, define for $x\in\mathfrak{A}_{\{j-1\}}$ and $y\in\mathfrak{A}_{\{j+1\}}$, $l_{\omega_j}(x):=\varphi(xP^j_{\omega_j})$, $r_{\omega_j}(x):=\varphi(P^j_{\omega_j}y)$.

The form l, r are positive functionals on $\mathfrak{A}_{\{j-1\}}$, $\mathfrak{A}_{\{j+1\}}$ respectively. Put

$$
\begin{aligned}
&H_j = 0\,, \widehat{H}_j = -\ln\rho_{\varphi\lceil\mathfrak{A}_{\{j\}}}\,, \quad \mathcal{R}(\varepsilon_j) = \mathbb{C}I\,;\\
&H_j = -\ln\rho_{\varphi\lceil\mathfrak{A}_{\{j\}}}\,, \widehat{H}_j = 0\,, \quad \mathcal{R}(\varepsilon_j) = \mathfrak{A}_{\{j\}}\,;\\
&H_j = -\sum_{\omega_j}\big(\ln\pi^j_{\omega_j}\big)P^j_{\omega_j}\,, \widehat{H}_j = 0\,, \quad \mathcal{R}(\varepsilon_j) = \mathfrak{A}_{\{j\},+}\,;\\
&H_{j,j+1} = -\ln\rho_{\varphi\lceil\mathfrak{A}_{\{j\}}}\,, \quad \mathcal{R}(\varepsilon_j) = \mathcal{R}(\varepsilon_{j+1}) = \mathbb{C}I\,;\\
&H_{j,j+1} = -\ln\rho_{\varphi\lceil\mathfrak{A}_{\{j+1\}}}\,, \quad \mathcal{R}(\varepsilon_j) = \mathbb{C}I\,, \mathcal{R}(\varepsilon_{j+1}) = \mathfrak{A}_{\{j+1\}}\,;\\
&H_{j,j+1} = -\sum_{\omega_{j+1}}\ln\rho_{l_{\omega_{j+1}}}P^{j+1}_{\omega_{j+1}}\,,\\
&\mathcal{R}(\varepsilon_j) = \mathbb{C}I\,, \mathcal{R}(\varepsilon_{j+1}) = \mathfrak{A}_{\{j+1\},+}\,;\\
&H_{j,j+1} = 0\,, \quad \mathcal{R}(\varepsilon_j) = \mathfrak{A}_{\{j\}}\,, \mathcal{R}(\varepsilon_{j+1}) = \mathbb{C}I\,;\\
&H_{j,j+1} = -\ln\rho_{\varphi\lceil\mathfrak{A}_{[j,j+1]}}\,, \quad \mathcal{R}(\varepsilon_j) = \mathfrak{A}_{\{j\}}\,, \mathcal{R}(\varepsilon_{j+1}) = \mathfrak{A}_{\{j+1\}}\,;\\
&H_{j,j+1} = -\sum_{\omega_{j+1}}\big(\ln\pi^{j+1}_{\omega_{j+1}}\big)P^{j+1}_{\omega_{j+1}}\,,\\
&\mathcal{R}(\varepsilon_j) = \mathfrak{A}_{\{j\}}\,, \mathcal{R}(\varepsilon_{j+1}) = \mathfrak{A}_{\{j+1\},+}\,;\\
&H_{j,j+1} = 0\,, \quad \mathcal{R}(\varepsilon_j) = \mathfrak{A}_{\{j\},+}\,, \mathcal{R}(\varepsilon_{j+1}) = \mathbb{C}I\,;\\
&H_{j,j+1} = -\sum_{\omega_j}\ln P^j_{\omega_j}\rho_{r_{\omega_j}}\,,\\
&\mathcal{R}(\varepsilon_j) = \mathfrak{A}_{\{j\},+}\,, \mathcal{R}(\varepsilon_{j+1}) = \mathfrak{A}_{\{j+1\}}\,;\\
&H_{j,j+1} = -\sum_{\omega_j,\omega_{j+1}}\big(\ln\pi^j_{\omega_j\omega_{j+1}}\big)P^j_{\omega_j}P^{j+1}_{\omega_{j+1}}\,,\\
&\mathcal{R}(\varepsilon_j) = \mathfrak{A}_{\{j\},+}\,, \mathcal{R}(\varepsilon_{j+1}) = \mathfrak{A}_{\{j+1\},+}\,.
\end{aligned}
\tag{4.2}
$$

Such opertors are even, and it is easy to verify that they satisfy the following commutation relations

$$
\begin{aligned}
&[H_j, H_{j,j+1}] = 0, \quad [H_{j,j+1}, \widehat{H}_{j+1}] = 0\,,\\
&[H_j, \widehat{H}_j] = 0, \quad [H_{j,j+1}, H_{j+1,j+2}] = 0\,.
\end{aligned}
\tag{4.3}
$$

Theorem 4.1. *Let $\varphi \in \mathcal{S}(\mathfrak{A})$ be a (locally faithful) Markov state.*
Then the pointwise norm–limit

$$\lim_{\substack{k\to j_-\\ l\to j_+}} e^{-ith_{[k,l]}}ae^{ith_{[k,l]}}$$

exists and defines a one–parameter automorphisms group $t \mapsto \sigma_t$ on the CAR algebra $\mathfrak{A}$ which admits φ as a KMS state. Further, φ has a normal faithful extension on all of $\pi_\varphi(\mathfrak{A})''$. In particular, any Markov state is faithful.

Proof. We start by noticing that, for each $k \leq l$,

$$h_{[k,l]} = H_k + \sum_{j=k}^{l-1} H_{j,j+1} + \widehat{H}_l. \tag{4.4}$$

Here, $h_{[k,l]}$ is the potential of φ relative to the region $[k,l]$ according to (4.1), and the even selfadjoint operators H_k, $H_{j,j+1}$, $\widehat{H}_l$ are given in (4.2) and satisfy the commutation relations (4.3). Thanks to these properties, the cocycle $e^{ith_{[k-1,l+1]}}e^{-ith_{[k,l]}}$ commutes with each element $a \in \mathfrak{A}$ localized in $\mathfrak{A}_{[k+1,l-1]}$. Then $e^{-ith_{[k,l]}}ae^{ith_{[k,l]}}$ becomes asymptotically constant (t fixed) on the localized elements $a \in \mathfrak{A}$, that is it trivially converges, pointwise in norm, on the localized elements of $\mathfrak{A}$. Next, by a standard 3–ϵ trick, it converges on all of $\mathfrak{A}$ and defines an isometry σ_t. It is straigthforward to show that $t \mapsto \sigma_t$ is actually a group of automorphisms of $\mathfrak{A}$, which is also pointwise–norm continuous in t, that is a strongly continuous group of automorphisms of $\mathfrak{A}$. By constuction, φ is automatically a KMS state for σ_t at inverse temperature $\beta = -1$.[6] The last assertions follow by [16], Corollary 5.3.9, taking into account that $\mathfrak{A}$ is a simple C^*–algebra ([15], Proposition 2.6.17). □

5. SOME ILLUSTRATIVE EXAMPLES

In this section we describe some natural examples of Markov states on the CAR algebra. We consider the case $I = \mathbb{Z}$ for the index set I.

We start by considering the case when the range of the two–step conditional expectations ε_n are always equal to $\mathbb{C}I$, $\nu \in I$, or $\mathfrak{A}_{\{n\}}$, $\nu \in I$. In this situation, it is immediate to show (by Theorem 3.1 or by direct computation) that the Markov state φ is the one–step product state extension of its restrictions to one–site local algebras:

$$\varphi(x_k \cdots x_l) = \varphi\lceil_{\mathfrak{A}_{\{k\}}}(x_k) \cdots \varphi\lceil_{\mathfrak{A}_{\{l\}}}(x_l)\,.$$

In this situation, φ is translation–invariant iff $\varphi\lceil_{\mathfrak{A}_{\{n\}}} = \varphi\lceil_{\mathfrak{A}_{\{n+1\}}} \circ \alpha$, where α is the one–step (right) shift on the chain. The Hamiltonian, which does not contains interaction terms, is easily written taking into account that it is a one–step product state.

[6] For the definition of Kubo–Martin–Schwinger boundary condition, as well as its connections with operator algebras and its meaning in quantum statistical mechanics, see [16] and the references cited therein.

Consider the case when the range of the two–step conditional expectations ε_n are all equal to $\mathfrak{A}_{\{n\},+}$. Then, $\Gamma = \mathbb{Z}$ and $\mathcal{E}$ is the trace–preserving conditional expectation onto the maximal Abelian subalgebra $\mathfrak{D} \sim C(\Omega)$ generated by $P_1^n \equiv a_n a_n^+$ and $P_2^n \equiv a_n^+ a_n$, $n \in \mathbb{Z}$. Under our definition, if $x \in \mathfrak{A}$, $\mathcal{E}(x)$ is represented by a continuous complex–valued function on Ω. Hence, we obtain

$$\varphi(x) = \int_\Omega \mathcal{E}(x)(\omega)\mu(\mathrm{d}\omega)\,.$$

Notice that, in this situation, the Markov state under consideration is the diagonal lifting to all of $\mathfrak{A}$, of the classical Markov process on $\mathfrak{D}$ obtained by $\varphi\lceil_{\mathfrak{D}}$.

The Markov state is translation invariant iff the underlying Markov measure μ on $\Omega \equiv \prod_{\mathbb{Z}}\{1,2\}$ is translation invariant, that is iff the transition coefficients $\frac{\varphi(P_k^j P_l^{j+1})}{\varphi(P_k^j)} =: \pi_{\omega_j\omega_{j+1}}$ does not depend on $j \in \mathbb{Z}$, and the all distributions coefficients $\varphi(P^j_{\omega_j}) =: \pi_{\omega_j}$ at places j coincide with the unique stationary distribution for the primitive matrix $\pi := [\pi_{\omega_j\omega_{j+1}}]$. Such a Markov state is the natural generalization of the Ising model to the CAR algebra.

The Hamiltonian for this Ising–like example is easily written taking into account that it is a diagonal lifting of a classical Markov chain, see (4.2). We report it for the sake of convenience.

$$\begin{aligned}
&H_j = -\sum_{\omega_j}\big(\ln \pi^j_{\omega_j}\big)P^j_{\omega_j}\,, \quad \widehat{H}_j = 0\,,\\
&H_{j,j+1} = -\sum_{\omega_j,\omega_{j+1}}\big(\ln \pi^j_{\omega_j,\omega_{j+1}}\big)\,.P^j_{\omega_j}P^{j+1}_{\omega_{j+1}}\,,
\end{aligned}$$

Theorem 5.1. *The translation invariant Markov state in the situation when* $\mathcal{R}(\varepsilon_n) = \mathfrak{A}_{\{n\}}$, $n \in I$ *is exponentially mixing w.r.t. the spatial translations. Moreover, it is a factor state.*

Proof. Let $k \leq l < m \leq n$ and $x \in \mathfrak{A}_{[k,l]}$, $y \in \mathfrak{A}_{[m,n]}$, we compute, taking into account that the functions on Ω representing $\mathcal{E}(x)$, $\mathcal{E}(y)$, depend only

on variables localized in $[k,l]$, $[m,n]$ respectively,

$$\varphi(xy)=\sum_{\omega_k,\dots,\omega_n}\pi_{\omega_k}\pi_{\omega_k\omega_{k+1}}\cdots\pi_{\omega_{n-1}\omega_n}\mathcal{E}(x)(\omega_k,\dots,\omega_l)\mathcal{E}(y)(\omega_m,\dots,\omega_n)$$

$$=\sum_{\substack{\omega_k,\dots,\omega_l\\ \omega_m,\dots,\omega_n}}\pi_{\omega_k}\pi_{\omega_k\omega_{k+1}}\cdots\pi_{\omega_{l-1}\omega_l}(\pi^{m-l})_{\omega_l\omega_m}\pi_{\omega_m\omega_{m+1}}\cdots\pi_{\omega_{l-1}\omega_l}$$

$$\times\,\mathcal{E}(x)(\omega_k,\dots,\omega_l)\mathcal{E}(y)(\omega_m,\dots,\omega_n)$$

$$\to_{m-l\to+\infty}\sum_{\omega_k,\dots,\omega_l}\pi_{\omega_k}\pi_{\omega_k\omega_{k+1}}\cdots\pi_{\omega_{l-1}\omega_l}\mathcal{E}(x)(\omega_k,\dots,\omega_l)$$

$$\times\sum_{\omega_m,\dots,\omega_n}\pi_{\omega_m}\pi_{\omega_m\omega_{m+1}}\cdots\pi_{\omega_{n-1}\omega_n}\mathcal{E}(y)(\omega_m,\dots,\omega_n)=\varphi(x)\varphi(y)\,.$$

Here, the exponential rate of convergence follows taking into account that the r–power π^r of the primitive matrix π tends exponentially to the stochastic projection onto the one–dimentional subspace generated by the stationary distribution for π, see e.g. [23], Section I.9. The fact that the Markov state φ under consideration is a factor state, can be proved as follows. Namely, for $k=1,2,\dots$ define

$$k:=\begin{cases}2j+1\,, & j\geq 0\,,\ k\,\text{odd}\,,\\ -2j\,, & j<0\,,\ k\,\text{even}\,,\end{cases}$$

apply to the ordered set $k=1,2,\dots$ the construction of pag. 92 of [25] concerning the set $\big\{\{e_{mn}(k)\}_{m,n=1}^2\ \big|\ k=1,2,\dots\big\}$, and consider the new local structure generated by the algebras

$$\mathfrak{B}_{\{j\}}:=\text{span}\big\{e_{mn}(k(j))\ \big|\ m,n=1,2\big\}\,.$$

Obiously,

(i) $\mathfrak{A}_{\{j\},+}\subset\mathfrak{B}_{\{j\}}\,,\ j\in\mathbb{Z}$,
(ii) $\big[\mathfrak{B}_{\{j_1\}},\mathfrak{B}_{\{j_2\}}\big]=0\,,\ j_1\neq j_2\,,\ j_1,j_2\in\mathbb{Z}$,
(iii) $\overline{\bigvee_{j\in\mathbb{Z}}\mathfrak{B}_{\{j\}}}^{C^*}=\mathfrak{A}$.

The last assertion directly follows from Theorem 2.6.10 of [15], by applying the previous considerations about the clustering to the new filtration $\big\{\mathfrak{B}_{\{j\}}\big\}_{j\in\mathbb{Z}}$. □

Other interesting examples are the two–block factors (see [1] for the analogy with the classical situation). These (two) examples arise when the ranges of two–point expectations are alternatively $\mathbb{C}I$ and $\mathfrak{A}_{\{\,\cdot\,\}}$, say, $\mathcal{R}(\varepsilon_{2n})=\mathbb{C}I$ and $\mathcal{R}(\varepsilon_{2n+1})=\mathfrak{A}_{\{2n+1\}}$. In the last situation, we get

$$\varphi(x_{2k}x_{2k+1}\cdots x_{2l}x_{2l+1})=\varphi\lceil_{\mathfrak{A}_{[2k,2k+1]}}(x_{2k}x_{2k+1})\cdots\varphi\lceil_{\mathfrak{A}_{[2l,2l+1]}}(x_{2l}x_{2l+1})\,,$$

that is it is the two–point product state extension. It is two–step translation invariant iff $\varphi\lceil_{\mathfrak{A}_{\mathfrak{A}_{[2n,2n+1]}}}=\varphi\lceil_{\mathfrak{A}_{[2n+2,2n+3]}}\circ\alpha^2$, α being the shift on the chain.

The Hamiltonian for this two–block factor, which is a particular case of those described in (4.2), is easily written as follows.

$$H_{2j,2j+1}=-\ln\rho_{\varphi\lceil_{\mathfrak{A}_{[2j,2j+1]}}}\,,\quad H_{2j+1,2j+2}=0\,,$$
$$H_{2j}=\widehat{H}_{2j+1}=0\,,\quad H_{2j+1}=-\ln\rho_{\varphi\lceil_{\mathfrak{A}_{\{2j+1\}}}}\,,\quad \widehat{H}_{2j}=-\ln\rho_{\varphi\lceil_{\mathfrak{A}_{\{2j\}}}}\,.$$

The Hamiltonian for the other example of two–block factor is written in a similar way.

To end the section, the following remark is in order. By applying Theorem 5.1 and Proposition 3 of [22], it is immediate to show that all the other states considered in this section, as well as the family $\{\psi_\omega\circ\mathcal{E}\}_{\omega\in\Omega}$ appearing in (3.5) (equivalently in (3.7)), denoted symbolically by η, with GNS representation π_η, provide examples for which the *algebra at infinity* $\mathfrak{Z}^{\perp}_{\pi_\eta}$ is trivial.[7] One could conclude that the states η are factor states if he is able to prove the inclusion $\mathfrak{Z}_{\pi_\eta}\subset\pi_\eta(\mathfrak{A}_+)''$, see the remark after Proposition 4 of [22]. Unfortunately, the last inclusion is false in general, see [21].

6. CONSTRUCTION OF MARKOV STATES

In this section we are going to demonstrate concrete constructions of Markov states. In the sequel we will assume that for the index set $I=\mathbb{Z}_-$. According to Proposition 2.2 it is enough to construct a functional φ on $\mathfrak{A}$, which is a Markov state with respect to the quasi-conditional expectation $E_{n]}$.

By $\mathcal{E}_{n]}$ (resp. $\mathcal{E}_{[n}$) we denote $\mathcal{E}_{n]}:\mathfrak{A}\to\mathfrak{A}_{n]}$ (resp. $\mathcal{E}_{[n}:\mathfrak{A}\to\mathfrak{A}_{[n,0]}$), here $n\in\mathbb{Z}_-$, the canonical Umegaki conditional expectation with respect to the trace. Note that the existence of such expectations have been proven by Araki and Moriya in [12].

Let be given an even positive operator $w_0\in\mathfrak{A}_{\{0\},+}$ and a sequence of even operators $\{K_{n-1,n}\}\subset\mathfrak{A}_{[n-1,n],+}$

Definition 6.1. *We say that the sequence $\{K_{n-1,n}\}$ describes a sequence of conditional density amplitudes if it satisfies the following conditions*

(i) $\mathcal{E}_{n-1]}(K_{n-1,n}K^*_{n-1,n})=\mathrm{id}\quad n\leq-2;$
(ii) $\mathcal{E}_{-1]}(K_{-1,0}w_0K^*_{-1,0})=\mathrm{id}$;

(iii) $\mathcal{E}_{[n}(K^*_{n-1,n}K_{n-1,n})=\mathrm{id}\quad n\leq-1.$

Denote

$$\mathbf{K}_{n-1}=K_{n-1,n}\cdots K_{-1,0}w_0^{1/2}\quad \mathbf{K}_{n-1,k}=K_{n-1,n}\cdots K_{k-1,k},\quad n<k.$$

[7]See [15], Definition 2.6.4 for the definition of the algebra at infinity.

For $n \in \mathbb{Z}_-$ put

$$W_{[n,0]} = \mathbf{K}_n^* \mathbf{K}_n.$$

Define $E_{n]} : \mathfrak{A} \to \mathfrak{A}_{n]}$ as follows

$$E_{n]}(x) = \mathcal{E}_{n]}(\mathbf{K}_n x \mathbf{K}_n^*), \quad x \in \mathfrak{A}. \tag{6.1}$$

From Corollary 4.8 [12] we infer the following

Lemma 6.2. *For every $n \in \mathbb{Z}_-$ the equality holds*

$$\Theta E_{n]} = E_{n]} \Theta.$$

Since Umegaki conditional expectation is completely positive, therefore we have

Lemma 6.3. *The map $E_{n]}$ defined by* (6.1) *is completely positive.*

Recall that a family $\{F_{[n,0]}\}$, where $F_{[n,0]} \in \mathfrak{A}_{[0,n]}$, is called *projective* if

$$\mathcal{E}_{[n}(F_{[n-1,0]}) = F_{[n-1,0]} \tag{6.2}$$

is valid for all $n \leq -1$.

Lemma 6.4. *The family $\{W_{[n,0]}\}$ is a projective family of density matrices.*

Proof. Using (iii) of Def.6.1 we check (6.2):

$$\begin{aligned}
\mathcal{E}_{[n}(W_{[n-1,0]}) =& \mathcal{E}_{[n}(w_0^{1/2} K_{-1,0}^* K_{-2,-1}^* \cdots K_{n-1,n}^* K_{n-1,n} \cdots K_{-1,0} w_0^{1/2}) \\
=& w_0^{1/2} K_{-1,0}^* K_{-2,-1}^* \cdots \mathcal{E}_{[n}(K_{n-1,n}^* K_{n-1,n}) \cdots K_{-1,0} w_0^{1/2} \\
=& w_0^{1/2} K_{-1,0}^* K_{-2,-1}^* \cdots K_{n,n+1}^* K_{n,n+1} \cdots K_{-1,0} w_0^{1/2} \\
=& W_{[n,0]}.
\end{aligned}$$

Finally, condition (ii) of Def.6.1 implies that such $W_{[n,0]}$ is a density matrix. ∎

Let $\tau_{[k,n]}$ be the normalized trace on $\mathfrak{A}_{[k,n]}$. Define a functional on $\mathfrak{A}_{[n,0]}$ as follows

$$\varphi_{[n,0]}(x) = \tau_{[n,0]}(W_{[n,0]} x), \quad x \in \mathfrak{A}_{[n,0]}.$$

Then using a property of Umegaki conditional expectations we infer that

$$\varphi_{[n,0]}(x) = \tau_{[n,0]}(\mathbf{K}_n x \mathbf{K}_n^*) = \tau_{\{n\}}(\mathcal{E}_{\{n\}}(\mathbf{K}_n x \mathbf{K}_n^*)),$$

here $\mathcal{E}_{\{n\}} : \mathcal{A}_{n]} \to \mathcal{A}_{\{n\}}$ is a Umegaki conditional expectation. According to Theorem 4.7 [12] we have

$$\mathcal{E}_{n]} \upharpoonright_{\mathfrak{A}_{[n}} = \mathcal{E}_{\{n\}}$$

therefore

$$\varphi_{[n,0]}(x) = \tau_{\{n\}}(\mathcal{E}_{n]}(\mathbf{K}_n x \mathbf{K}_n^*))$$

According to Lemma 6.4 and (i)-(ii) of Def.6.1 we conclude that such functionals are compatible family of states. So we can extend such states

$\varphi_{[n,0]}$ to $\mathfrak{A}$, which is denoted by φ. From Lemma 6.2 we conclude that φ is Θ-invariant.

Theorem 6.5. *The functional φ is a Markov state.*

Proof. From (6.1) and properties of the Umegaki conditional expectations one can see that the maps $E_{n]}$ are quasi-conditional expectations with respect to the triple $(\mathfrak{A}_{n-1]}, \mathfrak{A}_{n]}, \mathfrak{A})$, and it is easy to check that $E_{n]}(\mathfrak{A}_{[n}) \subset \mathfrak{A}_{\{n\}}$.

Since the operators $\{K_{n-1,n}\}_{n\in\mathbb{N}}$ and w_0 are even, therefore that states $\{\varphi_{[n,0]}\}_{n\in\mathbb{N}}$ are even, hence also φ is so.

Denote by $\varphi_{[n,k]}$ the restriction of the state φ on $\mathfrak{A}_{[n,k]}$. Let us find a density of the this state. Using the evenness of $K_{m,m+1}$ and (i)-(iii) of Def.6.1 we get

$$\varphi_{[n,k]}(x_n \cdots x_k) = \tau_{[n,0]}(\mathbf{K}_n(x_n \cdots x_k)\mathbf{K}_n^*) = \tau_{[n,k]}(\mathbf{K}_{n,k} x_n \cdots x_k \mathbf{K}_{n,k}^*).$$

Now we are going to check the first condition of the item (iii) of Proposition 2.2. To show one it is enough to verify the following equality

$$\varphi_{[n,h-1]}(E_{h]}(x)) = \varphi_{[n,0]}(x), \quad x \in \mathfrak{A}_{[n,0]}$$

We have

$$\begin{aligned}
\varphi_{[n,h-1]}(E_{h]}(x)) &= \tau_{[n,h-1]]}(\mathbf{K}_{n,h-1} E_{h]}(x)\mathbf{K}_{n,h-1}^*) \\
&= \tau_{[n,h-1]}(\mathbf{K}_{n,h-1}\mathcal{E}_{h]}(\mathbf{K}_h x \mathbf{K}_h^*)\mathbf{K}_{n,h-1}^*) \\
&= \tau_{[n,h-1]}(\mathcal{E}_{h]}(\mathbf{K}_{n,h-1}\mathbf{K}_h x \mathbf{K}_h^* \mathbf{K}_{n,h-1}^*)) \\
&= \tau_{[n,h-1]}(E_{n]}(x)) \\
&= \tau_{\{n\}}(E_{n]}(x)) = \varphi_{[n,0]}(x).
\end{aligned}$$

Thus φ is a Markov state. □

From this theorem we infer that any sequence of conditional density amplitudes defines a Markov state. Therefore it is enough to construct such kind of sequence to produce some concrete examples of Markov states. Now we give certain examples of sequences of conditional density amplitudes.

Example 6.1. Let us denote $\tilde{\epsilon}_n = \mathcal{E}_{n-1]} \upharpoonright_{\mathfrak{A}_{[n-1,n]}}$ and $\epsilon_n = \mathcal{E}_{[n} \upharpoonright_{\mathfrak{A}_{[n-1,n]}}$.

Define a sequence of operators $\{B_n\} \subset \mathfrak{A}_{[n-1,n]}$ as follows

$$\begin{aligned}
B_{n-1,n}(\alpha,\beta,\gamma,\delta) =& \alpha a_{n-1}^* a_{n-1} a_n^* a_n + \beta a_{n-1} a_{n-1}^* a_n^* a_n \\
&+ \gamma a_{n-1}^* a_{n-1} a_n a_n^* + \delta a_{n-1} a_{n-1}^* a_n a_n^*,
\end{aligned} \tag{6.3}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. It is clear that from (6.3) we have that $\Theta_I(B_{n-1,n}) = B_{n-1,n}$ for all $I \subset \mathbb{Z}_-$. This means each operator $B_{n-1,n}$ is even for all

$n \le 0$. Therefore, the operator $D_{n-1,n}(h) = \exp(hB_{n-1,n})$, $h \in \mathbb{R}$, is positive and even for all $n \le 0$.

From (6.3) we can easily get the equality

$$(B_{n-1,n}(\alpha,\beta,\gamma,\delta))^k = B_{n-1,n}(\alpha^k,\beta^k,\gamma^k,\delta^k)$$

for all $k, n \ge 1$. Using this we infer

$$D_{n-1,n}(h) = B_{n-1,n}(e^{h\alpha}, e^{h\beta}, e^{h\gamma}, \epsilon^{h\delta}).$$

Since $\tilde{\epsilon}_n$ and ϵ_n are Umegaki conditional expectations, we have

$$\tilde{\epsilon}_n(D_{n-1,n}(h)) = \frac{1}{2}\bigg((e^{h\alpha}+e^{h\beta})a^*_{n-1}a_{n-1} + (e^{h\gamma}+e^{h\delta})a_{n-1}a^*_{n-1}\bigg).$$

$$\epsilon_n(D_{n-1,n}(h)) = \frac{1}{2}\bigg((e^{h\alpha}+e^{h\beta})a^*_{n}a_{n} + (e^{h\gamma}+e^{h\delta})a_{n}a^*_{n}\bigg);$$

Impose that

$$e^{h\alpha}+e^{h\beta} = e^{h\gamma}+e^{h\delta}.$$

Therefore, denote

$$\kappa = \frac{e^{h\alpha}+e^{h\beta}}{2}.$$

Whence we have

$$\tilde{\epsilon}_n(D_{n-1,n}(h)) = \kappa\,\mathrm{id} \qquad \epsilon_n(D_{n-1,n}(h)) = \kappa\,\mathrm{id}\,. \tag{6.4}$$

Put $w_0 = \mathrm{id}$ and

$$K_{n-1,n} = \frac{1}{\sqrt{\kappa}}D_{n-1,n}(h/2).$$

From (6.4) we can prove the following: for every $n \in \mathbb{Z}_-$ we have

$$\tilde{\epsilon}_n(K_{n-1,n}K^*_{n-1,n}) = \mathrm{id}\,.$$

Indeed

$$\begin{aligned}\tilde{\epsilon}_n(K_{n-1,n}K^*_{n-1,n}) &= \tilde{\epsilon}_n(\frac{1}{\sqrt{\kappa}}D_{n-1,n}(h/2)D_{n-1,n}(h/2)\frac{1}{\sqrt{\kappa}}) = \\ &= \frac{1}{\kappa}\tilde{\epsilon}_n(D_{n-1,n}(h)) = \mathrm{id}\,. \qquad (6.5)\end{aligned}$$

Using the same argument one can show

$$\epsilon_n(K^*_{n-1,n}K_{n-1,n}) = \mathrm{id} \quad \forall n \le -1.$$

So according to Theorem 6.5 we can construct a Markov state. Note that the constructed QMS can be interpreted as 'Fermion' Ising model, and it coincides with the second illustrative example of section 5.

Example 6.2. Define a sequence of operators $\{U_n\} \subset \mathfrak{A}_{[n-1,n]}$ as follows

$$U_n = a_{n-1}^* a_n + a_n^* a_{n-1}.$$

Put $V_{n-1,n} = \exp(hU_n/2)$, where $h \in \mathbb{R}$. It is clear that each $V_{n-1,n}$ is positive and even, since U_n is even for all $n \in \mathbb{Z}_-$.

Now according to Theorem 4.7 [12] $\tilde{\epsilon}_n$ has a form

$$\tilde{\epsilon}_n(a) = \mathcal{E}_n^{(2)} \mathcal{E}_n^{(1)}(a), \quad a \in \mathfrak{A}_{[n-1,n]},$$

where $\mathcal{E}_n^{(2)}$ and $\mathcal{E}_n^{(1)}$ are defined in [12].

Let us compute the powers of U_n. We have

$$U_n^2 = a_{n-1}^* a_{n-1} a_n a_n^* + a_{n-1} a_{n-1}^* a_n^* a_n =: p_{n-1,n} + q_{n-1,n}$$

where we have denoted

$$p_{n-1,n} = a_{n-1}^* a_{n-1} a_n a_n^*, \quad q_{n-1,n} = a_{n-1} a_{n-1}^* a_n^* a_n.$$

It is easy to see that they are projections such that $p_{n-1,n} \cdot q_{n-1,n} = 0$. This implies that

$$U_n^{2k} = U_n^2, \quad k \geq 1$$

and therefore

$$U_n^{2k+1} = U_n, \quad k \geq 1$$

Then for $h \in \mathbb{R}$ we have

$$\begin{aligned}
\exp(hU_n) = \sum_{k\geq 0} \frac{h^k}{k!} U_n^k &= \sum_{k\geq 0} \frac{h^{2k}}{(2k)!} U_n^{2k} + \sum_{k\geq 0} \frac{h^{2k+1}}{(2k+1)!} U_n^{2k+1} \\
&= \mathrm{id} + \sum_{k\geq 1} \frac{h^{2k}}{(2k)!} U_n^2 + \sum_{k\geq 0} \frac{h^{2k+1}}{(2k+1)!} U_k \\
&= \mathrm{id} + (\sin h) U_n + (\cos h - 1) U_n^2
\end{aligned}$$

This implies that

$$\mathcal{E}_n^{(1)}(\exp(hU_n)) = (\exp(hU_n) + \Theta(\exp(hU_n)))/2 = \mathrm{id} + (\cos h - 1)U_n^2,$$

from this we get

$$\begin{aligned}
\tilde{\epsilon}_n(V_{n-1,n}^2) = \tilde{\epsilon}_n(\exp(hU_n)) &= \mathrm{id} + (\cos h - 1)\mathcal{E}_n^{(2)}(U_n^2) \\
&= \mathrm{id} + \frac{\cos h - 1}{2}(a_{n-1}^* a_{n-1} + a_{n-1} a_{n-1}^*) \\
&= \frac{\mathrm{id} + \cos h}{2} \mathrm{id}
\end{aligned}$$

Put $w_0 = \mathrm{id}$ and

$$K_{n-1,n} = \frac{1}{\sqrt{\alpha}} V_{n-1,n},$$

where $\alpha = \frac{1+\cos h}{2}$.

Using the argument of (6.5) one can prove the following equalities:

$$\tilde{\epsilon}_n(K_{n-1,n}K^*_{n-1,n}) = \mathrm{id} \quad \forall n \leq 0$$
$$\epsilon_n(K^*_{n-1,n}K_{n-1,n}) = \mathrm{id} \quad \forall n \leq -1.$$

So according to Theorem 6.5 we can construct a Markov state.

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